Linear Algebra - Final Exam

January 18, 2020 15.00:00-18.00:00 (plus extension in some cases) Online exam

Read carefully the exam rules on the Nestor page of the exam
Remember we will only grade exams uploaded at or before the deadline (counted down to the second). You are not allowed to use the internet during the exam. The only exception is if you use an online text editor (e.g., overleaf) to write the exam. But notice that you are not allowed to use an online editor if it has tools to solve math problems.

QUESTIONS:

1. **2** A system of equations with 3 unknowns (x, y, z), with parameters α and β, is given by

$$\begin{aligned} 2x + y - 3z &= \beta \\ x + \alpha z &= -2 \\ x + y &= 3\beta \end{aligned}$$

   (a) 0.6 Solve the system for $(\alpha, \beta) = (2, 1)$. Clearly indicate the applied steps. (You can use any correct method.)

   (b) 0.4 Use the determinant to determine for which value(s) of α and/or β the system has exactly one solution.

   (c) 1 Use row reduction to determine for which value(s) of α and/or β the system has no solutions.

2. **2** A linear transformation $L_1(\mathbf{x})$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ rotates a vector $\mathbf{x} \in \mathbb{R}^3$ by an angle ϕ around the x_3 axis, where ϕ is positive when the rotation is counterclockwise, that is when you rotate the vector in the direction from the x_1 axis towards the x_2 axis. A second transformation $L_2(\mathbf{x})$ from $\mathbb{R}^3$ to $\mathbb{R}^2$ projects a vector $\mathbf{x} \in \mathbb{R}^3$ onto the (x_1, x_2) plane.

   (a) 0.4 Calculate the matrix A_1 of the transformation $L_1(\mathbf{x})$.

   (b) 0.4 Calculate the matrix A_2 of the transformation $L_2(\mathbf{x})$.

   Both for (a) and (b) you need to explain how you got the matrix. (It is not enough to just write the correct matrix.)

   (c) 0.4 Calculate the single matrix A of the combined transformation in which one first applies L_1 to $\mathbf{x}$ and then L_2 to the result of $L_1(\mathbf{x})$.

   (d) 0.8 Find the eigenvalues and eigenvectors of A_1.

Please turn over

3. **2** Given a generic matrix $A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(a) 0.6 Find its eigenvalues in terms of the elements of A, a_{ij}.

Hint: If you choose things carefully, the characteristic polynomial will be naturally factored into the product of a first- and a second-order polynomial. Write the roots of those 2 polynomials in terms of the a_{ij}, noticing that $a_{13} = a_{23} = 0$, as given in the matrix.

(b) 0.6 Suppose that one of the eigenvalues of the previous matrix is 6, the trace of the matrix is $\mathrm{Tr}\,(A) = 16$, and the determinant of the 2×2 block of the matrix A given by $A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\det(A_1) = 21$. Find the other 2 eigenvalues of A.

(c) 0.8 Find all the elements of the matrix A if a set of three eigenvectors of A are:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where the order of the eigenvectors corresponds to the eigenvalues you found in (b) sorted such that $\lambda_1 < \lambda_2 < \lambda_3$.

4. **2** A coupled system of differential equations is given by

$$\begin{aligned} x_1'(t) &= 2\,x_1(t) + 3\,x_2(t) \\ x_2'(t) &= -2\,x_1(t) + 6\,x_2(t) \end{aligned}$$

with boundary conditions $x_1(0) = 4,\ x_1(0) = 1$.

(a) 1 Compute the eigenvalues and eigenvectors of the matrix of this system.

(b) 1 Use the eigenvalues and eigenvectors to compute the (real-valued) solution of the initial-value problem.

Note: The eigenvalues/eigenvectors have the square roots of some number, so don't be discouraged if you get that.

5. **2** Given the vector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}$ in $\mathbb{R}^3$, with a and b being any two real numbers.

(a) 0.3 Write the unit vector in the direction of the given vector.

(b) 0.3 Write the vector $\mathbf{v}_1$ in (a), when $a = 1$ and $b = 2$, and find the unit vector in $\mathbb{R}^3$ in the direction of this vector.

(c) 0.4 Using the dot product, find the general vector, $\mathbf{v}_2$ in $\mathbb{R}^3$, different from the null vector, that is orthogonal to the vector in part (b). (*Note: There are infinite vectors, each with a different modulus, that are orthogonal to a given vector. Here we ask you to write the general form of that vector; start with components x_1, x_2 and x_3, to find the form of that general vector.*)

(d) 0.3 Find the unit vector in $\mathbb{R}^3$ in the direction of $\mathbf{v}_2$. (*Hint: For this you can choose one, any, of the infinite vectors that you found in the previous point.*)

(e) 0.3 Write down any vector in $\mathbb{R}^3$ in the direction of $\mathbf{v}_2$.

(f) 0.4 Use the cross product to find a vector perpendicular to both the vector from part (a) and (c)